VERY MANY TERM CLONES IN A VERY SMALL VARIETY

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Abstract. We give a nice example of a finitely based locally finite variety which has uncountably many term clones.

Term clones. Let $\mathfrak{A} = (A, \Omega)$ be a universal algebra. A *term function* is a function $f : A^n \to A$ (for some $n \in \{0, 1, 2, \ldots\}$) which is induced by a term. A *term clone of* $\mathfrak{A}$ is a set of term functions which contains all the projections $\pi^n_k : A^n \to A$ and is closed under composition (also called "superposition"). The *full term clone of* $\mathfrak{A}$ is the set of all term functions.

Let $\mathbb{V}$ be a variety, $F_{\mathbb{V}}$ the free algebra in $\mathbb{V}$ on countably many generators $\{x_1, x_2, \ldots\}$. A *term clone of* $\mathbb{V}$ is a term clone of $F_{\mathbb{V}}$; since term functions are induced by elements of $F_{\mathbb{V}}$, we can equivalently view a term clone of $\mathbb{V}$ as a subset S of $F_{\mathbb{V}}$ which contains all the generators and is closed under the following "substitution" operation:

(∗∗) Whenever $t(x_1, \ldots, x_n) \in S$, and $t_1, \ldots, t_n \in S$, then also $t(t_1, \ldots, t_n) \in S$.

Our variety. Ivan Chajda has asked whether there is a locally finite variety $\mathbb{V}$ (preferably: finitely based) which has uncountably many term clones. We give here a nice example[1] of such a variety.

Our language contains one binary operation symbol $*$ and two constant symbols p and 0. We write $a{*}b{*}c$ for $(a{*}b){*}c$.

The laws of our variety $\mathbb{V}$ are

$$0{*}x = x{*}0 = 0, \qquad x{*}y{*}z = x{*}z{*}y, \qquad x{*}(y{*}z) = 0, \qquad x{*}y{*}y = 0.$$

The free algebra: elements. Let $\mathtt{0}, \mathtt{p}, \mathtt{x}_1, \mathtt{x}_2, \ldots$ be distinct objects.

We will describe an algebra $\mathfrak{F} = (F, *, p, 0) \in \mathbb{V}$ containing all $\mathtt{x}_i$ (in fact, $\mathfrak{F}$ will be freely generated by the $\mathtt{x}_i$ in $\mathbb{V}$).

In addition to $\mathtt{0}$, the set F will contain the following distinct objects:

(1) Letters: $\mathtt{p}, \mathtt{x}_1, \mathtt{x}_2, \ldots$
(2) Words: A word is a pair $w = (x, Y)$, where x is a letter and Y is a finite nonempty set of letters. Instead of $w = (x, \{y_1, \ldots, y_k\})$ (with all y_i distinct), we write w also as the string $x\, y_1 \cdots y_k$. We have to keep in mind that two apparently different

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[1] Ágnes Szendrei has pointed out that in fact many such varieties are known; any minimal variety generated by a finite primal algebra (which has at least 3 elements) will have all the required properties.

strings such as $x\,y_1y_2$ and $x\,y_2y_1$ are only two different notations for the same word $(x, \{y_1, y_2\})$.

We define the *length* of $\mathtt{0}$ to be 0, the length of any letter is 1, and the length of any word $(x, \{y_1, \dots, y_k\})$ (with all y_i distinct) is $k + 1$.

The free algebra: operations.

- The constant symbols 0 and p are interpreted as the objects $\mathtt{0}$ and $\mathtt{p}$, respectively.
- The product is defined naturally as follows:
  - If $x = 0$ or $y = 0$, then $x{*}y = 0$.
  - If x and y are letters, then $x{*}y = x\,y$.
  - If y is a word, then $x{*}y = 0$.
  - If x is a word, say $x = x_0\,x_1 \cdots x_k$ with $k \geq 1$, and y is a letter, $y \in \{x_1, \dots, x_k\}$, then $x{*}y = 0$.
  - If x is a word, say $x = x_0\,x_1 \cdots x_k$ with $k \geq 1$, and y is a letter, but $y \notin \{x_1, \dots, x_k\}$, then $x{*}y = x_0\,x_1 \cdots x_k y$.

It is easy to check (case by case) that this operation yields an algebra in $\mathbb{V}$, and it is also straightforward to see that this algebra is free over $\{\mathtt{x}_1, \mathtt{x}_2, \dots\}$.

Local finiteness. The free algebra over the empty set has just 3 elements: $\{\mathtt{0}, \mathtt{p}, \mathtt{p} * \mathtt{p}\}$. The free algebra over one free generator $\mathtt{x}$ is the set

$$\{\mathtt{0},\ \mathtt{p},\ \mathtt{x},\ \mathtt{p\,p},\ \mathtt{p\,x},\ \mathtt{x\,p},\ \mathtt{x\,x},\ \mathtt{x\,xp} = \mathtt{x\,px},\ \mathtt{p\,px} = \mathtt{p\,xp}\}.$$

In general, the $\mathbb{V}$-free algebra over n elements has exactly $1 + (n + 1) \cdot 2^{n+1}$ elements, so $\mathbb{V}$ is locally finite.

Uncountably many term clones. For any set $A \subseteq \{1, 2, 3, \dots\}$, we let $S(A)$ consist of the element $\mathtt{0}$, plus the set of all words that start with p whose length is in A, i.e., all words of the form

$$\mathtt{p\,x}_{i_2} \cdots \mathtt{x}_{i_n} \qquad i_2, \dots, i_n \text{ are all distinct, and } n \in A$$

or

$$\mathtt{p\,px}_{i_3} \cdots \mathtt{x}_{i_n} \qquad i_3, \dots, i_n \text{ are all distinct, and } n \in A$$

Applying the operation $(**)$ to any element $w \in S(A)$ will either result in 0, or in a word or letter w' of the same length as w.

[Applying $(**)$ to a word of the form $w = x\,y_1 \cdots y_n$, where x is one of the letters $\mathtt{x}_i$, may of course change the length of w; but the first letter of any word in $S(A)$ is the constant letter $\mathtt{p}$, so all applications of $(**)$ to a word in $S(A)$ will either just rename variables, or produce 0 because of $x * (y * z) = 0$.]

So $S(A)$ is closed under the operation $(**)$. For $A \neq A'$ we have $S(A) \neq S(A')$. So each $S(A)$ induces a different clone of term functions.

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